Derived geometric Satake equivalence, Springer correspondence, and small representations

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ICRA XVII

August 15, 2016

Motivation

Notation:

1. G - connected, reductive algebraic group over $\mathbb{C}$
2. $\check{G}$ - its Langlands dual group
3. W - the Weyl group of G (and of $\check{G}$)

Can we understand the functor:

$$\Phi : \mathrm{Rep}(\check{G}) \to \mathrm{Rep}(W)$$

given by

$$V \mapsto V^{\check{T}} \otimes \varepsilon?$$

Outline

1. The affine Grassmannian and Lusztig's embedding $\mathcal{N}_{\mathbf{GL}_n} \hookrightarrow \mathrm{Gr}_{\mathbf{GL}_n}$
2. A kind of generalization of this embedding for other groups ($\pi : \mathcal{M} \to \mathcal{N}$)
3. Connections to representation theory
4. A derived version

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The affine Grassmannian

G - connected, reductive algebraic group over $\mathbb{C}$
$\mathfrak{K} = \mathbb{C}((t)) \supset \mathfrak{O} = \mathbb{C}[[t]]$

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Structure:

$$\{\text{connected components}\} \xleftrightarrow{1-1} X^*(Z(\check{G}))$$

$G(\mathfrak{O}) \circlearrowright \mathrm{Gr}$ by left translation, and there is a bijection

$$\{G(\mathfrak{O}) \text{ - orbits}\} \xleftrightarrow{1-1} (X_*(T))^+$$

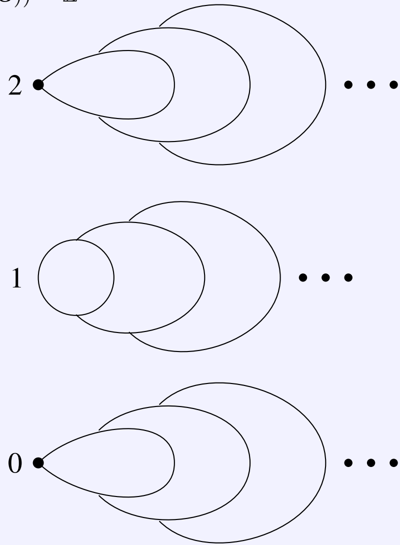
with

$$\mathrm{Gr}^{\check{\lambda}} \leftarrow\!\shortmid \check{\lambda}.$$

$\dim \mathrm{Gr}^{\check{\lambda}} = \langle 2\rho, \check{\lambda} \rangle$.

Example for $G = \mathbf{GL}_2(\mathbb{C})$, $\check{G} = \mathbf{GL}_2(\mathbb{C})$

$X^*(Z(\check{G})) = \mathbb{Z}$



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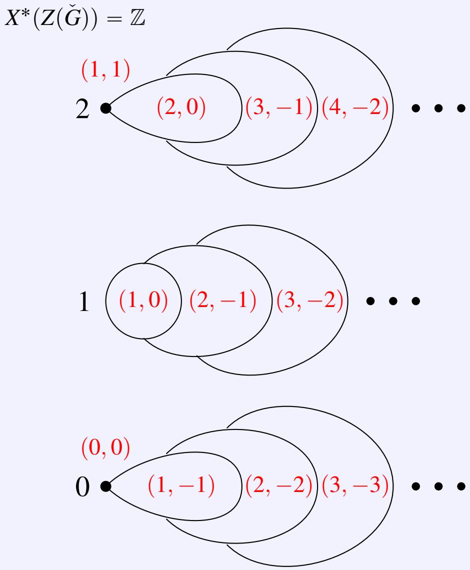

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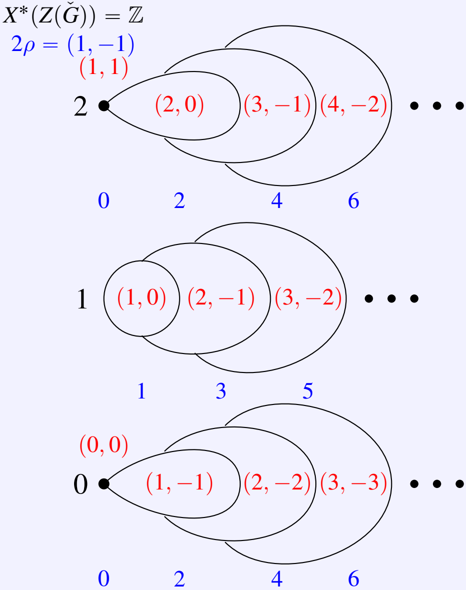

Lusztig's embedding $\mathcal{N}_{\mathbf{GL}_n} \hookrightarrow \mathrm{Gr}_{\mathbf{GL}_n}$

"Space of lattices"

Definition

A *lattice* is an $\mathfrak{O}$-submodule of $\mathfrak{K}^n$ of rank n.

$L_0 = \mathfrak{O}^n$ - standard lattice

As sets,

$$\text{space of lattices} = G(\mathfrak{K})/G(\mathfrak{O}).$$

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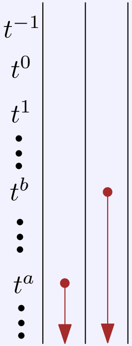
Using explicit computations with lattices, Lusztig gave

$$\mathcal{N}_{\mathbf{GL}_n} \hookrightarrow \{\text{lattices of valuation } 0 \text{ contained in } t^{-1}L_0\}.$$

Lusztig's embedding $\mathcal{N}_{\mathbf{GL}_n} \hookrightarrow \mathrm{Gr}_{\mathbf{GL}_n}$

A dominant weight (a, b) gives a lattice

$$(a, b) \quad \longmapsto$$



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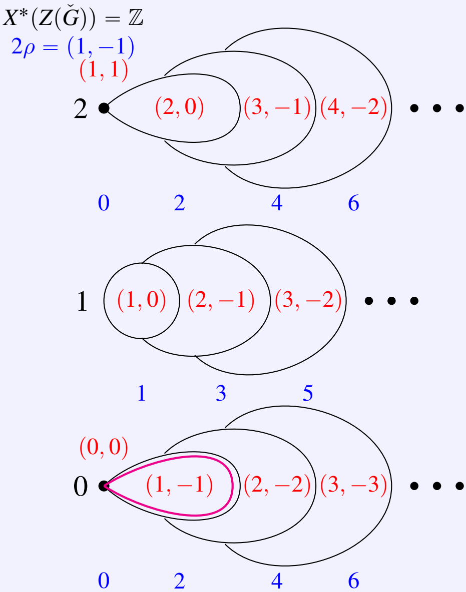

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$$
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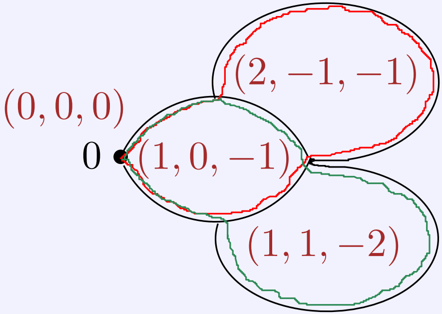
Dually,

$$\begin{array}{ccc} \mathcal{N}_{\mathbf{GL}_n} & \hookrightarrow & \{\text{lattices of valuation 0 containing } tL_0\} \\ \| & & \| \\ \mathcal{M}_2 & \hookrightarrow & \mathrm{Gr}_2^{\mathrm{sm}} \end{array}$$

Here, $\mathcal{M}_1 = \mathcal{M}_2$, but not true for $\mathbf{GL}_3(\mathbb{C})$.

Example for $G = \mathbf{GL}_3(\mathbb{C})$



Here, $\mathcal{M}_1 \neq \mathcal{M}_2$.

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Small representations

Definition

A *small representation* of $\check{G}$ has all of its weights in the root lattice and are such that their convex hull does not contain twice a root.

Let $\mathrm{Gr}^{\mathrm{sm}}$ be the closed subvariety of Gr that is the union of the $G(\mathfrak{O})$-orbits corresponding to small representations.

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Example

Let G = $\mathbf{SL}_2(\mathbb{C})$. Then $\check{G} = \mathbf{PGL}_2(\mathbb{C})$.

Dominant weights for $G = 0, 1, 2, \ldots$

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$\mathrm{Gr}^{\mathrm{sm}} = \mathrm{Gr}^0 \bigsqcup \mathrm{Gr}^2$

A kind of generalization of Lusztig's embedding to other groups

Define $\mathfrak{O}^- := \mathbb{C}[t^{-1}] \subset \mathfrak{K}$ and $\mathrm{Gr}_0^- := G(\mathfrak{O}^-) \cdot \mathbf{o}$. Define the open subvariety

$$\mathcal{M} := \mathrm{Gr}^{\mathrm{sm}} \cap \mathrm{Gr}_0^-$$

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Theorem (Achar–Henderson 2013)

There is a finite G-equivariant map

$$\pi : \mathcal{M} \to \mathcal{N}.$$

This gives a functor

$$\Psi = \pi_* \circ j^! : \mathrm{Perv}_{G(\mathfrak{O})}(\mathrm{Gr}^{\mathrm{sm}}) \to \mathrm{Perv}_G(\mathcal{N}).$$

Progress on understanding Φ

$$\begin{array}{ccc} \mathrm{Perv}_{G(\mathfrak{O})}(\mathrm{Gr}^{\mathrm{sm}}) & \qquad & \mathrm{Rep}(\check{G}) \\ \downarrow \Psi & & \downarrow \Phi \\ \mathrm{Perv}_G(\mathcal{N}) & & \mathrm{Rep}(W) \end{array}$$

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Theorem (Lusztig 1983)

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Theorem (Mirković–Vilonen 2007)

There is an equivalence of categories

$$\mathcal{S} : (\mathrm{Perv}_{G(\mathfrak{O})}(\mathrm{Gr}), *) \xrightarrow{\sim} (\mathrm{Rep}(\check G), \otimes)$$

$$\mathrm{IC}(\mathrm{Gr}^{\check\lambda}) \mapsto V(\check\lambda).$$

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The Springer correspondence

Nilpotent orbits in $\mathfrak{sl}_n$	Partitions of n	Irred. Reps. of $\mathfrak{S}_n$
Sizes of Jordan blocks	$\lambda = (\lambda_1, \ldots \lambda_n)$	$V(\lambda)$

It would be nice to bypass the combinatorics and directly relate representation theory to geometry.

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Let $\mu : \widetilde{\mathcal{N}} \to \mathcal{N}$ be the Springer resolution, and define the Springer sheaf

$$\mathrm{Spr} := \mu_* \underline{\mathbb{C}}_{\widetilde{\mathcal{N}}}[\dim \mathcal{N}] \in \mathrm{Perv}_G(\mathcal{N}).$$

Theorem (Springer, Lusztig, Borho–MacPherson)

W acts on Spr*, and there is a functor*

$$\mathbb{S} : \mathrm{Perv}_G(\mathcal{N}) \to \mathrm{Rep}(W)$$

given by

$$\mathcal{F} \mapsto \mathrm{Hom}(\mathrm{Spr}, \mathcal{F}).$$

The Springer correspondence

1. If $\mathcal{F}$ is simple, then $\mathbb{S}(\mathcal{F})$ is either simple or zero.
2. Thus, we have a bijection:

$$\text{subset of the simples in } \mathrm{Perv}_G(\mathcal{N}) \xleftrightarrow{1-1} \mathrm{Irr}(W)$$

The relationship between the two

Theorem (Achar–Henderson 2013, Achar–Henderson–Riche 2015)

The following diagram commutes:

$$\begin{array}{ccc} \mathrm{Perv}_{G(\mathfrak{O})}(\mathrm{Gr}^{\mathrm{sm}}) & \xrightarrow[\sim]{\mathcal{S}^{\mathrm{sm}}} & \mathrm{Rep}(\check{G})_{\mathrm{sm}} \\ \Psi \downarrow & & \downarrow \Phi \\ \mathrm{Perv}_G(\mathcal{N}) & \xrightarrow[\mathbb{S}]{} & \mathrm{Rep}(W) \end{array}$$

where $\Phi = (-)^{\check{T}} \otimes \varepsilon$.

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Extension to the mixed, derived setting

Theorem (M.)

Consider the following diagram.

$$\begin{array}{ccc} \mathcal{D}^{\mathrm{b,mix}}_{G(\mathfrak{O})}(\mathrm{Gr}^{\mathrm{sm}}) & \overset{\sim}{\xrightarrow[\text{[}\textit{Bez.}-\textit{Fink.}\text{ 2008]}]{\mathrm{der}\mathcal{S}^{\mathrm{sm}}}} & \mathcal{D}^{\mathrm{b}}\mathrm{Coh}^{\check{G}\times\mathbb{G}_m}(\check{\mathfrak{g}}^*)_{\mathrm{sm}} \\ \Psi \big\downarrow & & \big\downarrow \mathrm{der}\Phi \\ \mathcal{D}^{\mathrm{b,mix}}_{G}(\mathcal{N}) & \xrightarrow[\text{[}\textit{Rider}\text{ 2013]}]{\mathrm{der}\mathbb{S}} & \mathcal{D}^{\mathrm{b}}\mathrm{Coh}^{W\times\mathbb{G}_m}(\check{\mathfrak{h}}^*) \end{array}$$

There is a natural isomorphism of functors

$$\mathrm{der}\Phi \circ \mathrm{der}\mathcal{S}^{\mathrm{sm}} \Longleftrightarrow \mathrm{der}\mathbb{S} \circ \Psi$$

making the diagram commute.

A glimpse of the proof

It suffices to prove commutativity of the following diagram of additive categories

$$\begin{array}{ccc} \mathrm{Semis}_{G(\mathfrak{O})}(\mathrm{Gr}^{\mathrm{sm}}) & \xrightarrow[\sim]{\mathrm{der}\mathcal{S}^{\mathrm{sm}}} & \mathrm{Coh}_{\mathrm{fr}}^{\check{G}\times\mathbb{G}_m}(\check{\mathfrak{g}}^*)_{\mathrm{sm}} \\ \Psi\downarrow & & \downarrow \mathrm{der}\Phi \\ \mathrm{Semis}_G(\mathcal{N}) & \xrightarrow[\mathrm{der}\mathbb{S}]{} & \mathrm{Coh}_{\mathrm{fr}}^{W\times\mathbb{G}_m}(\check{\mathfrak{h}}^*) \end{array}$$

where $\mathcal{F} \in \mathrm{Semis}_{G(\mathfrak{O})}(\mathrm{Gr}^{\mathrm{sm}})$ is of the form

$$\mathcal{F} \simeq \mathrm{IC}(\mathrm{Gr}^{i_1})[n_1] \oplus \ldots \oplus \mathrm{IC}(\mathrm{Gr}^{i_m})[n_m].$$

and $\mathrm{der}\Phi = (- \otimes \varepsilon) \circ (-)^{\check{T}} \circ (- \otimes_{\mathcal{O}_{\check{\mathfrak{g}}^*}} \mathcal{O}_{\check{\mathfrak{h}}^*})$.

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and $\mathrm{der}\Phi = (- \otimes \varepsilon) \circ (-)^{\check{T}} \circ (- \otimes_{\mathcal{O}_{\check{\mathfrak{g}}^*}} \mathcal{O}_{\check{\mathfrak{h}}^*})$.
The proof goes in two steps:

1. Prove commutativity of the diagram for all groups of semisimple rank 1.
2. Show that each functor in the diagram commutes with restriction to a Levi subgroup of semisimple rank 1.

The End

Thanks!

Elaborate on Step 1

Reduces to $G = \mathbf{PGL}_2(\mathbb{C}), \check{G} = \mathbf{SL}_2(\mathbb{C})$.

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[Yun–Zhu 2011], [Lusztig 1995]

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